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Bulk-induced boundary perturbations

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Abstract

The influence of closed string moduli on the D-brane moduli space is studied from a worldsheet point of view. Whenever a D-brane cannot be adjusted to an infinitesimal change of the closed string background, the corresponding exactly marginal bulk operator ceases to be exactly marginal in the presence of the brane. The bulk perturbation then induces a renormalization group flow on the boundary whose end-point describes a conformal D-brane of the perturbed theory. We derive the relevant renormalization group equations in general and illustrate the phenomenon with a number of examples, in particular the radius deformation of a free boson on a circle. At the self-dual radius we can give closed formulae for the induced boundary flows which are exact in the boundary coupling constants.

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1. Introduction

The problem of how to stabilize the moduli of phenomenologically interesting string backgrounds is currently one of the central questions in string theory (for recent reviews, see [1, 2]). Most backgrounds of interest involve D-branes, and thus there are two kinds of moduli to consider: the D-brane moduli that describe the different D-brane configurations in a given closed string background and the closed string moduli that characterize this closed string background. Obviously, these two moduli spaces are not independent of one another; the moduli space of D-branes depends on the closed string background, and thus on the closed string moduli. On the other hand, the D-branes ‘back-react’ on the background, and thereby modify the original closed string background in which they were placed. In order to make progress with stabilizing all moduli in string theory, it is therefore of some significance to understand the interplay between these two moduli spaces better.

In this paper we make a small step towards this goal. It is well known that the closed string moduli space is described, in conformal field theory, by the exactly marginal bulk perturbations. A necessary condition for a bulk field to be exactly marginal is that it has

conformal weight $(1, 1)$ and that its three-point self-coupling vanishes [3, 4]. This condition was derived for conformal field theories without boundary, but in the presence of a D-brane, the situation changes. Indeed, a marginal bulk operator that is exactly marginal in the bulk theory may cease to be exactly marginal in the presence of a boundary.

The simplest example where this phenomenon occurs is the theory of a single free boson compactified on a circle. For this theory the full moduli space of conformal D-branes is known [5, 6] (see also [7, 8]). It depends in a very discontinuous manner on the radius of the circle, which is one of the bulk moduli. We always have the usual Dirichlet and Neumann branes, but if the radius is a rational multiple of the self-dual radius, the moduli space contains in addition a certain quotient of $SU(2)$. On the other hand, for an irrational multiple of the self-dual radius the additional part of the moduli space is just a line segment. The bulk operator that changes the radius is exactly marginal for the bulk theory, but in the presence of certain D-branes it is not. In particular, it ceases to be exactly marginal if we consider a rational multiple of the self-dual radius and a D-brane which is neither Dirichlet nor Neumann, but is associated with a generic group element g of $SU(2)$. If we change the radius infinitesimally, it is generically not a rational multiple of the self-dual radius any more, and thus the brane associated with g is no longer conformal.

In order to understand the response of the system to the bulk perturbation, we set up the renormalization group (RG) equations for bulk and boundary couplings. This can be done quite generally; we find that whenever certain bulk–boundary coupling constants do not vanish, the exactly marginal bulk perturbation is not exactly marginal in the presence of a boundary, but rather induces a non-trivial RG flow on the boundary. In particular, this therefore gives a criterion for when an exactly marginal bulk deformation is also exactly marginal in the presence of a boundary.

For the above example of the free boson, the resulting RG flow equations can actually be studied in quite some detail. We find that upon changing the radius, the resulting flow drives the brane associated with a generic group element g (that only exists at rational radii) to a superposition of pure Neumann or Dirichlet branes (that always exist). Whether the end-point is Dirichlet or Neumann depends on the sign of the perturbation, i.e. on whether the radius is increased or decreased. At the self-dual radius, the theory is equivalent to the $SU(2)$ WZW model at level 1, and the analysis can be done very elegantly. In this case, we can actually give a closed formula for the boundary flow which is exact in the boundary coupling (at first order in the bulk coupling).

Some of these results can be easily generalized to arbitrary current–current deformations of WZW models at higher level and higher rank. While we cannot, in general, give an explicit description of the whole flow any more, we can still describe at least qualitatively the end-point of the boundary RG flow.

The paper is organized as follows. In section 2 we derive the renormalization group equations that mix bulk and boundary couplings. In section 3 we apply these techniques to the free boson at the self-dual radius, and find the exact RG flow. Section 4 discusses how these results can be generalized to other rational radii as well as to current–current deformations of WZW models of higher level and rank. We conclude in section 5.

2. The renormalization group equation

In this section we shall analyse the RG flow involving bulk and boundary couplings. Bulk perturbations by relevant operators for conformal field theories with boundaries have been considered before in the context of integrable models starting from [9], and were further developed in [10–12]. In particular, these flows have been studied using (an appropriate

version of) the thermodynamic Bethe ansatz (see e.g. [13–17]), in terms of the truncated conformal space approach (see e.g. [15, 16, 18]), and recently by a form factor expansion [19, 20].

Let S^* be the action of a conformal field theory on the upper half plane. We denote the bulk fields by ϕ_i and the boundary fields by ψ_j . Their operator product expansions are of the form

$$\phi_i(z)\phi_j(w) = |z - w|^{h_k - h_i - h_j} C_{ijk} \phi_k(w) + \dots, \quad (2.1)$$

$$\psi_i(x)\psi_j(y) = (x - y)^{h_k - h_i - h_j} D_{ijk} \psi_k(y) + \dots, \quad (2.2)$$

where C_{ijk} and D_{ijk} are the bulk and boundary OPE coefficients, respectively. (For a general introduction to conformal field theory, see for example [21].) We are interested in the perturbation of this theory by bulk and boundary fields:

$$S = S^* + \sum_i \tilde{\lambda}_i \int \phi_i(z) d^2z + \sum_j \tilde{\mu}_j \int \psi_j(x) dx. \quad (2.3)$$

Introducing the length scale ℓ , we define dimensionless coupling constants λ_i and μ_j by

$$\tilde{\lambda}_i = \lambda_i \ell^{h_{\phi_i} - 2}, \quad \tilde{\mu}_j = \mu_j \ell^{h_{\psi_j} - 1}. \quad (2.4)$$

Note that we do not assume here that ϕ_i and ψ_j are marginal operators. If we expand the free energy in powers of λ_i and μ_j , we get terms of the form

$$\frac{\lambda_1^{l_1} \dots \mu_1^{m_1} \dots}{l_1! \dots m_1! \dots} \prod_i \ell^{(h_{\phi_i} - 2)l_i} \prod_j \ell^{(h_{\psi_j} - 1)m_j} \times \int \langle \phi_1(z_1^1) \phi_1(z_2^1) \dots \phi_2(z_1^2) \dots \psi_1(x_1^1) \dots \rangle \prod d^2z_k^i \prod dx_k^j. \quad (2.5)$$

To regularize (2.5), we use an UV cutoff ℓ . More precisely, the prescription is

$$|z_k^i - z_{k'}^{i'}| > \ell, \quad |x_k^j - x_{k'}^{j'}| > \ell, \quad \text{Im } z > \frac{\ell}{2}. \quad (2.6)$$

The parameter ℓ thus appears in (2.5) both explicitly as powers in h and implicitly through the range of integration.

Following [4] we now consider a change of the scale ℓ , $\ell \rightarrow (1 + \delta t)\ell$, and ask how the coupling constants have to be adjusted so as to leave the free energy unchanged. The explicit dependence of expression (2.5) on ℓ leads to a change in λ_i and μ_j by

$$\begin{aligned} \lambda_i &\rightarrow (1 + (2 - h_{\phi_i})\delta t)\lambda_i, \\ \mu_j &\rightarrow (1 + (1 - h_{\psi_j})\delta t)\mu_j. \end{aligned} \quad (2.7)$$

The implicit dependence of (2.5) on ℓ through the UV prescription (2.6) gives rise to an additional change of the coupling constants. From the first inequality in (2.6), which controls the UV singularity in the bulk operator product expansion, we obtain the equation $\delta\lambda_k = \pi C_{ijk} \lambda_i \lambda_j \delta t$ [4]. A similar calculation gives $\delta\mu_k = D_{ijk} \mu_i \mu_j \delta t$ (see, for example, [22]) for the contribution from the boundary operator product expansion (the second inequality). Finally, we have to consider the contribution from the third inequality which controls the singularity that arises when a bulk operator approaches the boundary. When we scale ℓ by $(1 + \delta t)$, we change the integration region of a bulk operator by a strip parallel to the real axis of width $\ell\delta t/2$. This changes expression (2.5) by terms of the form

$$- \lambda_i \ell^{h_{\phi_i} - 2} \int dx \int_{\ell/2}^{\ell/2 + \ell\delta t/2} dy \langle \dots \phi_i(z) \dots \rangle, \quad (2.8)$$

where we have written $z = x + iy$. In order to evaluate this contribution, we use the bulk–boundary operator product expansion

$$\phi_i(z, \bar{z}) = (2y)^{h_{\psi_j} - h_{\phi_i}} B_{ij} \psi_j(x) + \dots, \quad (2.9)$$

where B_{ij} is the bulk–boundary OPE coefficient that depends on the boundary condition in question. The change of the free energy described by (2.8) is then

$$-\lambda_i \ell^{h_{\phi_i} - 2} \int dx \frac{\ell \delta t}{2} B_{ij} \ell^{h_{\psi_j} - h_{\phi_i}} \langle \dots \psi_j(x) \dots \rangle = -\frac{1}{2} B_{ij} \ell^{h_{\psi_j} - 1} \lambda_i \delta t \int dx \langle \dots \psi_j(x) \dots \rangle, \quad (2.10)$$

which can be absorbed by a shift of $\delta \mu_j = \frac{1}{2} \lambda_i B_{ij} \delta t$. Collecting all terms, we thus obtain the RG equations to lowest order

$$\dot{\lambda}_k = (2 - h_{\phi_k}) \lambda_k + \pi C_{ijk} \lambda_i \lambda_j + \mathcal{O}(\lambda^3), \quad (2.11)$$

$$\dot{\mu}_k = (1 - h_{\psi_k}) \mu_k + \frac{1}{2} B_{ik} \lambda_i + D_{ijk} \mu_i \mu_j + \mathcal{O}(\mu \lambda, \mu^3, \lambda^2). \quad (2.12)$$

The flow of the bulk variables λ_k in (2.11) is independent of the boundary couplings μ_k on the disc. The RG flow in the bulk therefore does not depend on the boundary condition whereas the bulk has a significant influence on the flow of the boundary couplings. Note that the terms we have written out explicitly are independent of the precise details of the UV cutoff (if the fields are marginal). Higher order corrections, on the other hand, will depend on the specific regularization scheme.

Now suppose that ϕ_i is an exactly marginal bulk perturbation. The perturbation by ϕ_i is then exactly marginal in the presence of a boundary if the bulk–boundary coupling constants B_{ik} vanish; this has to be the case for all boundary fields ψ_k (except for the vacuum) that are relevant or marginal, i.e. satisfy $h_{\psi_k} \leq 1$. Obviously, switching on the vacuum on the boundary just leads to a rescaling of the disc amplitude; for irrelevant operators, on the other hand, the flow is damped by the first term of (2.12), and thus the bulk perturbation only leads to a small correction of the boundary condition.

The above condition is the analogue of the usual statement about exact marginality: a necessary condition for a marginal bulk (boundary) operator to be exactly marginal is that the three point couplings C_{iik} (D_{iik}) vanish for all marginal or relevant fields ϕ_k (ψ_k), except for the identity (see, for example, [3, 4, 8]).

If the bulk–boundary coefficient B_{ik} does not vanish for some relevant or marginal boundary operator ψ_k , the corresponding boundary coupling μ_k starts to run, and there is a non-trivial RG flow on the boundary. The bulk couplings λ_i are not affected by the flow ($\dot{\lambda}_i = 0$), and we can thus interpret it as a pure boundary flow in the marginally deformed bulk model. From that point of view, it is then clear that the flow must respect the g -theorem [22, 23]. In particular, the g -function of the resulting brane is smaller than that of the initial brane. This is in fact readily verified for the examples we are about to study.

3. The free boson theory at the self-dual radius

As an application of these ideas, we now consider the example of the free boson theory at $c = 1$. We shall first consider the theory at the critical radius, where it is in fact equivalent to the WZW model of $su(2)$ at level 1. For this theory all conformal boundary states are known [24], and are labelled by group elements $g \in SU(2)$ (for earlier work, see also [25, 26]).

Suppose that we are considering the boundary condition labelled by $g \in SU(2)$, where we write

$$g = \begin{pmatrix} a & b^* \\ -b & a^* \end{pmatrix}, \quad (3.1)$$

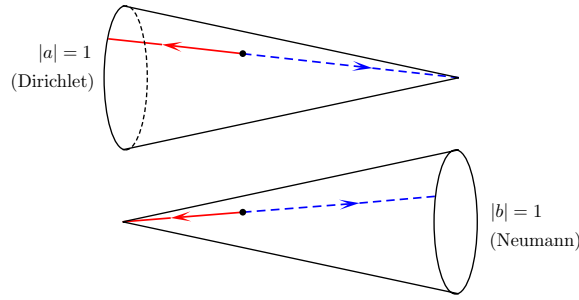


Figure 1. The moduli space of D-branes on the self-dual circle, $SU(2)$, can be described as a product of two circles S^1 (given by the phases of a and b in (3.1)) fibred over an interval where $|a|$ runs between 0 and 1, and $|a|^2 + |b|^2 = 1$. The ends of the interval where one of the circles shrinks to zero describe Dirichlet and Neumann branes, respectively. If we start with a generic boundary condition and increase (decrease) the radius, the boundary condition will flow to a Dirichlet (Neumann) boundary condition.

and a and b are complex numbers satisfying $|a|^2 + |b|^2 = 1$. (Geometrically, $SU(2)$ can be thought of as a product of two circles; see figure 1.) We shall choose the convention that the brane labelled by g satisfies the gluing condition¹

$$(gJ_m^\alpha g^{-1} + \bar{J}_{-m}^\alpha) \|g\rangle = 0, \quad (3.2)$$

where J^α are the currents of the WZW model (the corresponding Lie algebra generators will be denoted by t^α). We shall furthermore use the identification that g diagonal ($b = 0$) describes a Dirichlet brane on the circle, whose position is given by the phase of a ; conversely, if g is off-diagonal ($a = 0$), the brane is a Neumann brane, whose Wilson line on the dual circle is described by the phase of b .

3.1. Changing the radius

We want to consider the bulk perturbation by the field

$$\Phi = J^3 \bar{J}^3, \quad \text{where} \quad t^3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3.3)$$

This is an exactly marginal bulk perturbation that changes the radius of the underlying circle. With the above conventions, the perturbation $\lambda\Phi$ with $\lambda > 0$ increases the radius, while $\lambda < 0$ decreases it. At any rate, the perturbation by Φ breaks the $su(2)$ symmetry down to $u(1)$. However, in the presence of a boundary, the bulk perturbation is generically not exactly marginal any more. This is implicit in the results of [5–7] since the set of possible conformal boundary conditions is much smaller at a generic (irrational) radius relative to the self-dual case. Here we want to study in detail what happens to a generic boundary condition under this bulk deformation.

Even before studying the detailed RG equations that we derived in the previous section, it is not difficult to see that the above deformation is generically not exactly marginal. In particular, we can consider the perturbed one-point function of the field Φ in the presence of the boundary. To first order, this means evaluating the two-point function

$$\lambda \int_{\mathbb{H}^+} d^2z \langle (J^\alpha \bar{J}^\alpha)(z) (J^\alpha \bar{J}^\alpha)(w) \rangle, \quad (3.4)$$

¹ Note that the labelling differs from the one used in [5].

where the label $\alpha = 3$ is not summed over. Using the usual doubling trick [27] this amplitude can be expressed as a chiral four-point function, where we have the fields J^α at z and w and the ‘reflected’ fields $J^\beta \equiv gJ^\alpha g^{-1}$ at \bar{z} and \bar{w} .

The chiral correlation functions of WZW models at level k can be calculated using the techniques of [28, 29]. Let t^α , $\alpha = 1, \dots, \dim(g)$, be the Lie algebra generator (corresponding to J^α) in some representation; we choose the normalization

$$\text{Tr}(t^\alpha t^\beta) = k\delta^{\alpha\beta}. \quad (3.5)$$

To evaluate $\langle J^{\alpha_1}(z_1) \cdots J^{\alpha_n}(z_n) \rangle$, consider then all permutations $\rho \in S_n$ that have no fixed points; this subset of permutations is denoted by \tilde{S}_n . Each such ρ can be written as a product of disjoint cycles

$$\rho = \sigma_1 \sigma_2 \dots \sigma_M. \quad (3.6)$$

To each cycle $\sigma = (i_1 i_2 \dots i_m)$, we assign the function

$$f_\sigma^{\alpha_{i_1} \dots \alpha_{i_m}}(z_{i_1}, \dots, z_{i_m}) = -\frac{\text{Tr}(t^{\alpha_{i_1}} \dots t^{\alpha_{i_m}})}{(z_{i_1} - z_{i_2})(z_{i_2} - z_{i_3}) \cdots (z_{i_m} - z_{i_1})}, \quad (3.7)$$

and to each ρ the product $f_{\sigma_1} \dots f_{\sigma_M}$. The correlation function is then given by summing over all permutations without fixed points,

$$\langle J^{\alpha_1}(z_1) \cdots J^{\alpha_n}(z_n) \rangle = \sum_{\rho \in \tilde{S}_n} f_\rho. \quad (3.8)$$

In (3.4), ρ is either a 4-cycle or consists of two 2-cycles. In the latter case, we get the terms

$$\frac{(\text{Tr}(t^\alpha t^\beta))^2}{|z - \bar{z}|^2 |w - \bar{w}|^2} + \frac{(\text{Tr}(t^\alpha t^\beta))^2}{|z - \bar{w}|^4} + \frac{\text{Tr}(t^\alpha t^\alpha) \text{Tr}(t^\beta t^\beta)}{|z - w|^4}. \quad (3.9)$$

Integration over the upper half plane gives (divergent) contributions proportional to $|w - \bar{w}|^{-2}$, which can be absorbed in the renormalization of J^α . The six terms that come from six different 4-cycles give a total contribution of

$$-\frac{\text{Tr}([t^\alpha, t^\beta]^2)}{(z - \bar{z})(w - \bar{w})|z - \bar{w}|^2}. \quad (3.10)$$

Set $w = i|w|$ and $z = x + iy$. The resulting integral over the upper half plane is logarithmically divergent for $y \rightarrow 0$. Introducing an ultraviolet cutoff ϵ , we get

$$\int_{\mathbb{R}} dx \int_{\epsilon}^{\infty} dy \frac{1}{2iy} \frac{1}{2i|w|} \frac{1}{x^2 + (y + |w|)^2} = \frac{\pi}{4|w|^2} \log \epsilon - \frac{\pi}{8|w|^2} \log |w|^2 + \mathcal{O}(\epsilon). \quad (3.11)$$

The first term has the right w dependence to be absorbed by a suitable renormalization of J^α . The second term, however, pushes the conformal weight away from $(1, 1)$. Thus, if J^α is to be exactly marginal, the expression $\text{Tr}([t^\alpha, t^\beta]^2)$ must vanish.

In the case above, $\text{Tr}([t^\alpha, t^\beta]^2)$ equals

$$\text{Tr}([t^3, g t^3 g^{-1}]^2) = -8|a|^2 |b|^2. \quad (3.12)$$

This only vanishes if either $|a| = 0$ or $|b| = 0$; the corresponding boundary conditions are therefore either pure Dirichlet or pure Neumann boundary conditions. This ties in with the expectations based on the analysis of the conformal boundary conditions since only pure Neumann or Dirichlet boundary conditions exist for all values of the radius.

The argument above can also be used in the general case to derive a necessary criterion for when a bulk deformation is exactly marginal in the presence of a boundary. It is not difficult to see that it leads to the same criterion as the one given in section 2.

3.2. The renormalization group analysis

Now we want to analyse what happens if g does not describe a pure Neumann or pure Dirichlet boundary condition. In particular, we can use the results of section 2 to understand how the system reacts to the bulk perturbation by $\lambda\Phi$.

In order to see how the boundary theory is affected by the perturbation, we have to compute the bulk–boundary OPE of the perturbing field Φ . There are no relevant boundary fields (except the vacuum), and the marginal fields are all given by boundary currents J^ν . We can thus determine the bulk–boundary OPE coefficient $B_{\Phi\nu}$ from the two-point function

$$\langle J^\nu(x)(J^3\bar{J}^3)(z) \rangle = B_{\Phi\nu}|z - \bar{z}|^{-1}|x - z|^{-2}, \quad (3.13)$$

which—employing the general formula (3.8)—leads to

$$B_{\Phi\nu} = -i \text{Tr}(t^\nu[t^3, gt^3g^{-1}]). \quad (3.14)$$

We see that the only boundary field that is switched on by the bulk perturbation is the current J^ν whose (Hermitian) Lie algebra generator t^ν is proportional to the commutator $[t^3, gt^3g^{-1}]$. The normalized t^ν is given by

$$t^\nu = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -e^{i\chi} \\ e^{-i\chi} & 0 \end{pmatrix} \quad \text{with} \quad ab^* = |ab|e^{i\chi}. \quad (3.15)$$

Its relation to the commutator is

$$-i[t^3, gt^3g^{-1}] = -i \begin{pmatrix} 0 & -2ab^* \\ 2a^*b & 0 \end{pmatrix} = Bt^\nu, \quad (3.16)$$

where the bulk–boundary coefficient $B = B_{\Phi\nu}$ is given by

$$B = -2\sqrt{2}|a||b|. \quad (3.17)$$

The boundary current proportional to t^ν modifies the boundary condition g by

$$\delta g = it^\nu g = \frac{1}{\sqrt{2}} \begin{pmatrix} -a\frac{|b|}{|a|} & b^*\frac{|a|}{|b|} \\ -b\frac{|a|}{|b|} & -a^*\frac{|b|}{|a|} \end{pmatrix}. \quad (3.18)$$

This leaves the phases of a and b unmodified, but decreases the modulus of a while increasing that of b .

Since the operators are marginal, the renormalization group equation to lowest order in the coupling constants (2.12) is now

$$\dot{\mu} = \frac{1}{2}B\lambda + \mathcal{O}(\mu\lambda, \mu^2, \lambda^2), \quad (3.19)$$

where μ is the boundary coupling constant of the field J^ν . Thus if the radius is increased ($\lambda > 0$), μ becomes negative, and the boundary condition flows to the boundary condition with $b = 0$; the resulting brane is then a Dirichlet brane whose position is determined by the original phase of a . Conversely, if the radius is decreased ($\lambda < 0$), μ becomes positive, and the boundary condition flows to the boundary condition with $a = 0$. The resulting brane is then a Neumann brane whose Wilson line is determined by the original value of the phase of b (see figure 1). This is precisely what one should have expected since for radii larger than the self-dual radius, only the Dirichlet branes are stable, while for radii less than the self-dual radius, only Neumann branes are stable.

Actually, the renormalization group flow can be studied in more detail. It follows from (3.18) that to lowest order in μ

$$a(\mu) = a_0 - \mu a_0 \frac{|b_0|}{\sqrt{2}|a_0|} + \mathcal{O}(\mu^2), \quad (3.20)$$

where the initial values of a and b have been denoted by a_0 and b_0 , respectively. Since a depends on the RG parameter only via μ , it thus follows that

$$\dot{a} = -\mu a \frac{|b|}{\sqrt{2}|a|} = -\frac{B}{2\sqrt{2}} \frac{|b|}{|a|} a \lambda = |b|^2 a \lambda = (1 - |a|^2) a \lambda. \quad (3.21)$$

If we write $|a| = \sin \psi$, this simplifies to

$$\dot{\psi} = \sin \psi \cos \psi \lambda. \quad (3.22)$$

Denoting the RG parameter by t , the solution to this differential equation is

$$\tan \psi(t) = \tan \psi(0) e^{\lambda t}. \quad (3.23)$$

Thus for $\lambda > 0$ this flows indeed to $|a_\infty| = 1$, while for $\lambda < 0$ we find $|a_\infty| = 0$, as expected.

Given relation (3.20), we can deduce from the solution for $a(t)$ a differential equation for $\mu(t)$ which turns out to be

$$\dot{\mu} = -\sqrt{2} \dot{\psi}. \quad (3.24)$$

This can be integrated to

$$\mu(t) = -\sqrt{2}(\psi(t) - \psi(0)). \quad (3.25)$$

We can thus determine the path on the group manifold as

$$g(t) = e^{i\mu(t)t^\gamma} g. \quad (3.26)$$

As a consistency check, one verifies that

$$\lim_{t \rightarrow \infty} g(t) = \begin{cases} \begin{pmatrix} \frac{a}{|a|} & 0 \\ 0 & \frac{a^*}{|a|} \end{pmatrix} & \text{if } \lambda > 0 \\ \begin{pmatrix} 0 & \frac{b^*}{|b|} \\ -\frac{b}{|b|} & 0 \end{pmatrix} & \text{if } \lambda < 0. \end{cases} \quad (3.27)$$

The path is actually a geodesic on $SU(2)$, relating the point g to the nearest diagonal or off-diagonal group element. In order to see this, we write

$$g = \begin{pmatrix} \sin \psi e^{i\theta} & \cos \psi e^{-i\varphi} \\ -\cos \psi e^{i\varphi} & \sin \psi e^{-i\theta} \end{pmatrix}, \quad (3.28)$$

where $0 \leq \psi \leq \frac{\pi}{2}$ and $0 \leq \theta, \varphi < 2\pi$. In these variables, the metric on $SU(2)$ is

$$ds^2 = d\psi^2 + \sin^2 \psi d\theta^2 + \cos^2 \psi d\varphi^2. \quad (3.29)$$

The above path in $SU(2)$ is the path with θ and φ constant. The variable μ (see equation (3.25)) is simply proportional to $\psi - \psi_0$, which is the arc length parameter along the curve.

4. Generalizations

It is not difficult to generalize the above analysis in a number of different ways.

4.1. The free boson away from criticality

If the radius of the free boson is a rational multiple of the self-dual radius, $R = \frac{M}{N} R_{\text{sd}}$, then a similar analysis applies. At this radius, the conformal boundary states are labelled by elements

in the quotient space

$$g \in SU(2)/\mathbb{Z}_M \times \mathbb{Z}_N, \quad (4.1)$$

where \mathbb{Z}_M and \mathbb{Z}_N act by multiplication by roots of unity on a and b , respectively, leaving the absolute values unaffected [5]. One way to arrive at this construction is to describe the theory at radius R as a freely acting orbifold by $\mathbb{Z}_M \times \mathbb{Z}_N$ of the self-dual radius theory [30]. Under this orbifold action none of the generic $SU(2)$ branes are invariant, and thus the branes of the orbifold are simply the superpositions of MN branes of the $SU(2)$ level 1 theory.

In particular, it therefore follows that the bulk–boundary OPE coefficients that were relevant in the above analysis are (up to an MN -dependent factor) unmodified. Therefore, the same conclusions as above hold: if the radius is increased, a generic brane flows to M equally spaced Dirichlet branes (this is the interpretation of the branes with $b = 0$); if the radius is decreased, a generic brane flows to N Neumann branes whose Wilson lines are equally spaced on the dual circle (i.e. the branes with $a = 0$). Since the phases of a and b are unchanged along the flow, the flow is obviously compatible with the $\mathbb{Z}_M \times \mathbb{Z}_N$ orbifold operation that only acts on these phases.

4.2. The analysis at higher level

For $SU(2)$ at level k , the branes that preserve the affine symmetry (up to an inner automorphism by conjugation by a group element $g \in SU(2)$) are labelled by $\|j, g\rangle\rangle$, where $j = 0, \frac{1}{2}, 1, \dots, \frac{k}{2}$ denotes the different representations of $\hat{su}(2)$ at level k (that label the different Cardy branes [31]), while g describes the automorphism

$$(g J_m^\alpha g^{-1} + \bar{J}_{-m}^\alpha) \|j, g\rangle\rangle = 0. \quad (4.2)$$

In addition, there is the identification

$$\|j, g\rangle\rangle = \|\frac{k}{2} - j, -g\rangle\rangle, \quad (4.3)$$

where $-g \in SU(2)$ is minus the 2×2 matrix (3.1).

The field Φ is an exactly marginal bulk field for any level k [32, 33]. We can thus ask what happens to the boundary condition $\|j, g\rangle\rangle$ as we perturb the theory by Φ .

In fact, it is easy to see that the above analysis for level 1 still goes through—the only place where k enters is in the overall normalization of the bulk–boundary OPE coefficient that is largely irrelevant for our analysis. Thus if we perturb the theory by the exactly marginal bulk perturbation $J^3 \bar{J}^3$, the brane labelled by $\|j, g\rangle\rangle$ flows to $\|j, g_0\rangle\rangle$, where g_0 is either diagonal or off-diagonal (depending on the sign of the bulk coupling constant λ), and the relevant phase of a_0 or b_0 agrees with the original phase of a or b in g , respectively. In particular, this prescription therefore respects the identification (4.3). It is also worth noting that it does not mix different j , and therefore does not produce any additional flows that would reduce the K-theoretic charge group [34, 35].

The bulk perturbation breaks the $SU(2)$ symmetry down to $SU(2)/U(1) \times U(1)$, where the radius of the $U(1)$ factor is deformed away from the original value of \sqrt{k} times the self-dual radius. The branes corresponding to g_0 (to which any brane will flow) describe factorizable boundary conditions that define a standard Dirichlet or Neumann boundary condition for the $U(1)$ factor. It is then clear that these branes exist for an arbitrary radius of this $U(1)$ (this has been analysed previously in [36, 37]). The resulting picture is therefore again in agreement with expectations.

For large values of the level k , we can give yet another geometric interpretation. The current–current deformation of the WZW model can be understood as deforming the metric,

the B-field and the dilaton on the group. In particular, once the WZW model is deformed, the dilaton ϕ is not constant any more, but has the dependence (see [33, 36, 38])

$$e^{-2\phi(\psi)} = \frac{1 - (1 - R^2) \cos^2 \psi}{R}, \quad (4.4)$$

where R denotes the deformed radius of the embedded $U(1)$ ($R = 1$ being the WZW case). If we start with a D0-brane on the group at position g , then after the deformation it will flow along the gradient of the dilaton to a maximum, such that its mass, which is proportional to $\frac{1}{g_s} \sim e^{-\phi}$, is minimal. Minimization of (4.4) leads to the conditions

$$(1 - R^2) \sin 2\psi = 0, \quad (1 - R^2) \cos 2\psi > 0. \quad (4.5)$$

When the radius is increased ($R > 1$, corresponding to $\lambda > 0$), we find $\psi = \frac{\pi}{2}$, i.e. $|a| = 1$. For $R < 1$ we obtain on the other hand $\psi = 0$ ($|b| = 1$). This is thus in nice agreement with our analysis of section 3.

4.3. Other bulk perturbations

So far we have only considered bulk perturbations by $J^3 \bar{J}^3$, but it should be clear how to generalize this to the case where the perturbing bulk field is $J^\alpha \bar{J}^{\bar{\alpha}}$. In fact, if we write $t^\alpha = ht^3 h^{-1}$ and $t^{\bar{\alpha}} = \bar{h}t^3 \bar{h}^{-1}$, then the above analysis goes through provided we replace g by $\hat{g} = h^{-1} g \bar{h}$. Indeed, the relevant t^γ is in this case

$$it^\gamma \propto [t^\alpha, g t^{\bar{\alpha}} g^{-1}] = h[t^3, \hat{g} t^3 \hat{g}^{-1}] h^{-1}, \quad (4.6)$$

and thus

$$\delta g = \delta(h \hat{g} \bar{h}^{-1}) = h \delta \hat{g} \bar{h}^{-1}. \quad (4.7)$$

At level 1, the perturbation by $J^\alpha \bar{J}^{\bar{\alpha}}$ can again be interpreted as changing the radius of a circle. Its embedding in $SU(2)$ is described as

$$\theta \mapsto h e^{i\theta t^3} \bar{h}^{-1}. \quad (4.8)$$

4.4. Higher rank groups

Much of the discussion for $SU(2)$ carries over to Lie groups of higher rank, though in general it is not possible to give a closed expression for the integrated flow any more. For simplicity we shall restrict the following discussion to the Lie groups $G = SU(n)$.

Let us consider a D-brane that is characterized by the gluing condition (3.2) for a given $g \in SU(n)$. As in section 3.1, the perturbation $J^\alpha \bar{J}^{\bar{\alpha}}$ with α fixed and $t^\alpha \in su(n)$ is exactly marginal in the bulk [32, 33], but leads to a flow of the gluing parameter g as

$$\dot{g} = \frac{\lambda}{2} [t^\alpha, t^\beta] g, \quad (4.9)$$

where $t^\beta = g t^\alpha g^{-1}$. This flow can be interpreted as a gradient flow,

$$\dot{g} = -\nabla V(g) \quad \text{with potential} \quad V(g) = -\frac{\lambda}{2} \text{Tr}(t^\alpha g t^\alpha g^{-1}). \quad (4.10)$$

To see this, we first recall that the gradient is defined by

$$\left. \frac{d}{ds} V(g + istg) \right|_{s=0} = -\text{Tr}(\nabla V(g) g^{-1} it), \quad (4.11)$$

where t is an arbitrary vector in the Lie algebra. Here, the minus sign appears because the trace is negative definite on the Lie algebra; the factors of g map it to a tangent vector itg at g , and the tangent vector $\nabla V(g)$ to an element of the Lie algebra, $\nabla V(g) g^{-1}$. Evaluating the

directional derivative, we find

$$\begin{aligned} \left. \frac{d}{ds} V(g + is t g) \right|_{s=0} &= -\frac{\lambda}{2} \text{Tr}(t^\alpha i t g t^\alpha g^{-1} - t^\alpha g t^\alpha g^{-1} i t) \\ &= \frac{\lambda}{2} \text{Tr}([t^\alpha, g t^\alpha g^{-1}] g (g^{-1} i t)). \end{aligned} \quad (4.12)$$

Comparing this with (4.11), we deduce that

$$\nabla V(g) = -\frac{\lambda}{2} [t^\alpha, g t^\alpha g^{-1}] g, \quad (4.13)$$

which hence implies that (4.10) reproduces the flow equation (4.9).

In contradistinction to the $SU(2)$ case, however, this flow is generically not a geodesic flow. The change of the direction of the RG flow is

$$\frac{d}{dt} [t^\alpha, t^\beta] \propto [t^\alpha, [t^\beta, [t^\alpha, t^\beta]]], \quad (4.14)$$

which is in general not proportional to $[t^\alpha, t^\beta]$. Thus, the tangent to the flow is not parallel to a fixed direction in the Lie algebra; this makes it hard to integrate the complete flow in the generic case.

We can nevertheless describe at least qualitatively the end point of the flow. To this end, it is sufficient to understand the fixed points of the flow and their stability properties.

A boundary condition corresponding to the gluing condition g is a fixed point of the flow if $[t^\alpha, t^\beta] = 0$. This is only the case if the matrices t^α and t^β have common eigenspaces. Assume that t^α is generic, i.e. that all its eigenvalues τ_i are distinct and all eigenspaces $\mathbb{R}v_i$ are one dimensional. Then $[t^\alpha, t^\beta] = 0$ if and only if g permutes the n eigenspaces and multiplies each one by a phase r_i . This means that there are $n!$ discrete choices for g , each coming with $n - 1$ continuous degrees of freedom (note that $\det g = \pm \prod_i r_i = 1$).

This has a simple physical interpretation if the level of the WZW model is 1. Then the theory is equivalent to a compactification on a torus described by the momentum lattice

$$\{(p_L, p_R) \in \Lambda_W \oplus \Lambda_W, p_L - p_R \in \Lambda_R\}, \quad (4.15)$$

where Λ_W and Λ_R are the weight and root lattice of $su(n)$, respectively. Without loss of generality, we may choose our Cartan subalgebra such that it contains t^α . A group element $g \in SU(n)$ that permutes the eigenvectors v_i acts by conjugation on the root lattice and hence corresponds to some element w_g of the Weyl group. The gluing condition (3.2) for the currents J^β then translates into the condition

$$w_g p_L = p_R \quad (4.16)$$

for the momenta. This is the gluing condition for the standard torus branes that couple to all momenta p_L (as $w_g p_L - p_L \in \Lambda_R$). The dimension of the brane is given by the number of eigenvalues of w_g that are not equal to 1 (this is the absolute length of w_g). The phases of g then correspond to the positions and Wilson lines of the brane.

These standard torus D-branes are the ones that are unaffected by a perturbation of the size of the torus and they correspond to the fixed points g of the flow equation (4.9).

In order to understand where a generic brane flows to, it is furthermore important to understand the stability of the fixed points. Suppose we start with a boundary condition that is very close to one of the fixed points; if the brane is driven back to the fixed point it is *stable*, if it flows away (to some other fixed point) it is *unstable*.

To simplify the discussion, we shall work in the eigenbasis $\{v_i\}$ of t^α . Using its spectral decomposition $t^\alpha = \sum \tau_i P_i$, we can rewrite (4.9) as

$$\dot{g} = \frac{\lambda}{2} \sum_{i,j} \tau_i \tau_j (P_i g P_j - g P_i g^{-1} P_j g). \quad (4.17)$$

To check the stability of a fixed point $g = S$, consider the ansatz

$$g_{ij}(t) = S_{ij} + \epsilon h_{ij}(t). \quad (4.18)$$

Here, S is the matrix of a fixed point given by a permutation σ and phases r_i , i.e.

$$S : v_i \mapsto r_i v_{\sigma(i)}. \quad (4.19)$$

In particular, this means that

$$SP_i S^{-1} = P_{\sigma(i)}. \quad (4.20)$$

Evaluating (4.17) to first order yields

$$\dot{h}_{ij} = \frac{\lambda}{2} (\tau_i - \tau_{\sigma(j)}) (\tau_j - \tau_{\sigma^{-1}(i)}) h_{ij}. \quad (4.21)$$

We easily see that $\dot{h}_{ij} = 0$ for $i = \sigma(j)$; these are the $n - 1$ flat directions we have identified before. In order for $g = S$ to be stable, all other components h_{lm} must have negative eigenvalues. Without loss of generality, we may assume that the τ_i are ordered,

$$\tau_1 < \tau_2 < \dots < \tau_n. \quad (4.22)$$

Consider then the coefficient for $i = \sigma(p)$. If $\lambda > 0$, the condition is

$$j < p \Rightarrow \sigma(j) < \sigma(p), \quad (4.23)$$

i.e. σ grows monotonically, which is only the case for $\sigma = \text{id}$. For $\lambda < 0$, σ must be a decreasing function, i.e.

$$\sigma : i \mapsto n - i. \quad (4.24)$$

We thus obtain a very simple result: if $\lambda > 0$, g flows to the identity component; if $\lambda < 0$, the D-brane flows to the component where g inverts the order of the eigenvalues of t^α .

In the torus picture (for $k = 1$), the identity component corresponds to the D0-branes. This is what we expect: if the size increases ($\lambda > 0$) beyond the self-dual radius, the D0-branes are the lightest branes and a generic brane will flow to one of them. If the size decreases ($\lambda < 0$), the physical intuition is less clear, because there is a B-field on the torus which complicates things. The torus branes which are described by the inverse ordering of the eigenvectors correspond to the longest element w_0 in the Weyl group². Its absolute length (minimal number of reflections or minimal number of transpositions) is given by $\lfloor \frac{n-1}{2} \rfloor$ which gives us the dimension of the D-brane on the torus. In the example of $SU(3)$, the branes which are stable under a perturbation with $\lambda < 0$ are thus D1-branes.

So far we have restricted our discussion to a generic perturbation t^α . It is clear that there are special directions t^α for which the bulk perturbation breaks less symmetry. If two or more eigenvalues of t^α coincide, one observes from (4.21) that there are more directions h_{ij} which are unaffected by the flow ($\dot{h}_{ij} = 0$), i.e. the dimensions of the moduli spaces of fixed points can grow beyond $n - 1$.

For other bulk perturbations $J^\alpha \bar{J}^{\bar{\alpha}}$ with $t^{\bar{\alpha}} \neq t^\alpha$, the discussion is very similar to the one above. Assume that $t^{\bar{\alpha}} = \bar{h} (\sum \bar{\tau}_i P_i) \bar{h}^{-1}$ with eigenvalues $\bar{\tau}_1 < \dots < \bar{\tau}_n$. Then the above arguments apply if we replace g by $\hat{g} = g \bar{h}$. If the level is 1, we again have an interpretation in terms of a torus in $SU(n)$ which is obtained from the Cartan torus by translation by \bar{h}^{-1} from the right.

For large values of the level k , we can—as in the $SU(2)$ case in section 4.2—interpret the perturbation as a deformation of the metric, the B-field and the dilaton on the group (see [39]). One would then expect that the group values to which the branes flow are again characterized by the property that they maximise the dilaton; it would be interesting to check this directly.

² Here ‘long’ refers to the standard length which is the minimal number of reflections at simple roots needed to write w_0 , or, in terms of permutations, the minimal number of transpositions of neighbouring elements.

5. Conclusions

In this paper we have studied the interplay between open and closed string moduli on the disc. In particular, we have shown that an exactly marginal closed string perturbation (that describes the change of a closed string modulus) may cease to be exactly marginal in the presence of a D-brane. If this is the case, the bulk operator induces a RG flow on the boundary. The end-point of the RG flow is a D-brane that is conformal in the perturbed bulk theory. We have illustrated this phenomenon with the example of the free boson theory at $c = 1$, and with current–current deformations of WZW models.

It would be interesting to analyse similar phenomena in a time-dependent string theory context. Suppose, for example, that we deform the bulk theory of some D-brane string background infinitesimally so that the D-brane is no longer conformal. One would then expect that the background evolves in a time-dependent process towards a configuration in which the D-brane is again conformal. Neglecting closed string radiation, time dependence is essentially incorporated by substituting the first-order derivatives in the RG equations by second-order time derivatives (see e.g. [40, 41]). Since the models we considered are compact, unlike the situation studied in [41] there is no open string radiation that could escape to infinity. In particular, there is therefore no dissipation and the model will undergo eternal oscillations. It would be interesting to study the effects of closed string radiation in the examples we considered above. In particular, by suitably controlling the bulk deformation λ , the process can be made arbitrarily slow.

Our analysis was originally motivated by trying to understand the interpretation of the obstruction of [42]. There $N = 2$ supersymmetric B-type D-branes on the orbifold line T^4/\mathbb{Z}_4 of K3 were studied using matrix factorization and conformal field theory techniques. It was found that a certain B-type brane (namely the brane that stretches diagonally across the two T^2 s that make up T^4) is obstructed against changing the relative radii of the two T^2 s; this could be seen both from the matrix factorization point of view, as well as in conformal field theory.

The analysis above suggests that upon changing the relative radii, the brane simply readjusts its angle so that it continues to stretch diagonally across the two tori. From the point of view of conformal field theory, there is no obstruction in this. The obstruction that was observed in the matrix factorization analysis only means that the resulting brane breaks the B-type supersymmetry, as could also be seen in conformal field theory [42]. It would be interesting to understand more directly when such a phenomenon may happen in conformal field theory; the relevant condition will probably be related to the charge constraint of [42].

At least in this example the obstruction therefore does not ‘lift’ the corresponding bulk modulus. While we have only analysed the disc amplitude, we do not expect any higher order corrections since the brane remains supersymmetric (albeit not B-type supersymmetric). In general, however, one would expect that the backreaction of the brane on the background geometry could lift bulk moduli. This backreaction is however not visible at the disc level, and one will have to analyse at least the annulus amplitudes in order to study it in conformal field theory. It would be very interesting to find a simple example where this can be analysed explicitly.

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